Last Solution Set, MA4027, Summer 2004

4.1.10 Find the smallest 3-regular graph with connectivity 1.

Solution: We must find the smallest 3-regular graph G with a cut vertex. Since G is 3-regular, we know that $n(G)$ is even. Suppose $n \leq 8$. If x is a cut vertex, then $G - x$ contains a component C with at most three vertices. For $y \in V(C)$, $d_C(y) \leq 2$, but then either $xy \in E(G)$ or $zy \in E$ for $x \neq z \in V(G) - V(C)$, in either case, x cannot be a cut vertex. So $n \geq 10$. For a ten-vertex three-regular graph with a cut vertex (two, actually), start with two copies of K_4. In each, subdivide one edge by adding a vertex. You now have eight vertices of degree three, and two of degree two. Insert an edge joining the two vertices of degree two. The graph is now 3-regular, and the vertices just joined by the additional edge are both cut vertices. Since no 3-regular graph with connectivity one can have fewer than ten vertices, and since the graph just constructed is 3-regular, has ten vertices, and has connectivity one, this is the smallest.

5.1.25 Let $G = (V, E)$ be the unit-distance graph in $\mathbf{R}^2$, with $V = \mathbf{R}^2$ and $xy \in E$ if and only if $d(x, y) = 1$ (this is Euclidean distance). Prove that $4 \leq \chi(G) \leq 7$.

Proof: We first prove the lower bound. Take two copies of $K_4 - e$, drawn so that every edge has unit length and regard these drawings as rigid objects. Call these G_1 and G_2. Starting with a vertex of degree two, apply the greedy coloring algorithm to each graph. The result will be a three-coloring of each in which the vertices of degree two have color 1 and the vertices of degree three are assigned colors 2 and 3. Let x_1 and x_2 be the vertices of degree two in G_1, and y_1, y_2 the vertices of degree two in G_2. Identify x_1 and y_1, to obtain a graph on seven vertices. The vertex obtained by identifying x_1 and x_2 has degree four, and the remaining vertex degrees are unchanged. Now carefully adjust the positions of x_2, y_2 so that $d(x_2, y_2) = 1$. We need a new color. Thus $\chi(G) \geq 4$.

To prove the upper bound, we construct a proper 7-coloring of the unit-distance graph. To get started, construct a hexagonal “honeycomb” tiling of the plane, using hexagons of diameter 1, oriented in such a way that each hexagon has two edges parallel to the horizontal axis. There are “diagonals” of hexagons running southwest to northeast and northwest to southeast, and columns of hexagons running vertically. Our pallette is the set $\{0, 1, \ldots, 6\}$. Choose a hexagon, and assign the color 0 to its interior and its lower edge (endpoints included). Having assigned a color k to the interior of a particular hexagon, assign $k + 1 \pmod 7$ to the interior of its neighbor to the northeast, $k - 1 \pmod 7$ to that of its neighbor to the southeast, $k + 3 \pmod 7$ to that of its neighbor above, and $k + 4 \pmod 7$ to that of its neighbor below. Horizontal bounding edges (and their endpoints) take the color of the region immediately above. Colors assigned to diagonal bounding edges can be taken from either of the adjacent regions. The result is a proper 7-coloring of the plane such that any two points x, y satisfying $d(x, y) = 1$ are assigned different colors. It follows that $\chi(G) \leq 7$. $\square$

5.1.26 Given finite sets $S_1, S_2, \ldots, S_m$, let $U = S_1 \times S_2 \times \cdots \times S_m$. Define $G = (U, E)$ by putting $u \leftrightarrow v$ iff u and v differ in every coordinate. Determine $\chi(G)$.

Solution: All vertices that agree in the ith coordinate are pairwise nonadjacent, and so can have the same color, so we can use the ith coordinate to partition U into i color classes. It follows that $\chi(G) = \min_i |S_i|$.

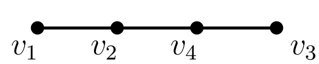
5.1.33 Prove that every graph G has a vertex ordering relative to which the greedy coloring algorithm uses exactly $\chi(G)$ colors.

Proof: Let G be any graph, and let f be a proper coloring of G using $\chi(G)$ colors. Furthermore, suppose that f has the property that the set of vertices receiving color j cannot be enlarged without changing the color of a vertex that received some color $k > j$. Number the vertices by color class, i.e., first label all vertices receiving color 1, then label all vertices using color 2, etc. If we now run the greedy algorithm using the constructed labeling, the algorithm will recreate f. $\square$

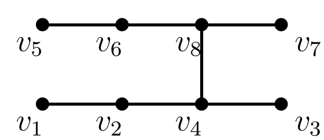
5.1.34 For each $k \in \mathbf{N}$, recursively define a tree T_k, of order 2^k and having maximum degree $\Delta(T_k) = k$, and an ordering α of $V(T_k)$ such that the greedy coloring with respect to α uses $k+1$ colors. Comment on the effectiveness of the greedy algorithm for arbitrary graphs.

Solution: We may take $T_0 = K_1$, with $1 = 2^0$ vertex and maximum degree $\Delta = 0$. Having constructed T_k $(k \geq 0)$, we construct T_{k+1} in the following way: we take two copies T_k^1 and T_k^2 of T_k. Label the vertices of T_k^1 and T_k^2 as $v_1, \ldots, v_k$ in such a way that the greedy algorithm is forced to use a $(k+1)$st color at each of v_k and v_{2k}. Now relabel each vertex of T_k^2 by adding k to its previous index, i.e. v_i is relabeled as v_{k+i}. The greedy algorithm is forced to use a $(k+1)$st color at v_k. Upon arriving at v_{2k}, it is then forced to use a $(k+2)$nd color at v_{2k}. The accompanying illustration shows T_2 and T_3.

T_2:



T_3:



This indicates that the performance of the greedy algorithm can be arbitrarily bad, since we can recursively construct 2-colorable graphs (trees are bipartite, hence 2-colorable) on which the greedy algorithm uses arbitrarily many colors.

5.1.38 Prove that $\chi(\overline{G}) = \omega(\overline{G})$ if G is bipartite.

Solution: Here are two proofs. Each shows that $\theta(G) = \alpha(G)$

Proof: Assume that $G \ncong K_{m,n}$, since otherwise the problem is trivial. It suffices to show that $\theta(G) = \alpha(G)$. Since an isolated vertex contributes exactly one to both $\theta(G)$ and $\alpha(G)$, we may assume that G has no isolated vertex. Suppose that K is a minimum-cardinality clique cover of G. Since G is bipartite and has no isolated vertices, every maximal clique of G is an edge, so θ is also a minimum-cardinality edge cover of G. By Corollary 3.1.24 ("König's other theorem"?) in West, $\theta(G) = \alpha(G)$, and the proof is complete. □

Proof: Assume that $G \ncong K_{m,n}$, since otherwise the problem is trivial. It suffices to show that $\alpha(G) = \theta(G)$. We know that $\alpha(G) \leq \theta(G)$, since a maximal clique contributes at most one vertex to any stable set. We proceed by construction. Let M be a maximum matching in G. Let U_X and U_Y denote the set of unsaturated vertices in X and Y, respectively, and let S denote the set of saturated vertices in X. Since M is maximal, it follows that $I = U_X + U_Y + S$ is a stable set, with cardinality $|I| = |U_X| + |U_Y| + |S|$. Moreover, $K = U_X + U_Y + M$ is a clique cover, since every vertex not in either U_X or U_Y is incident to an edge in M. Since $|S| = |M|$, then $|K| = |I|$, and the result follows. □

7.2.7 We have a $3 \times 3 \times 3$ block of cheese. A mouse wants to eat the entire thing, by starting at a corner, eating one 1×1 subcube at a time, starting a new subcube only if it shared a face with the subcube just completed, and completing the task by consuming the 1×1 subcube that was initially at the center. Can the mouse get his wish?

Solution: Nope. Color the subcubes using, say, black and white, so that the corner subcubes are black and such that subcubes with shared faces receive different colors. It is easy to see that the resulting coloring produces fourteen black and thirteen white subcubes. Think of it as a bipartite graph with fourteen black vertices and thirteen white vertices. The center vertex representing the subcube that the mouse wants to eat last is white. So the mouse plans to follow a Hamiltonian path from a black vertex to a white vertex, but such a path cannot exist unless the numbers of black and white vertices are equal. □

7.2.17 Prove that the Cartesian product of two Hamiltonian graphs is Hamiltonian. Conclude that the cube Q_k is Hamiltonian when $k \geq 2$.

Proof: Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be Hamiltonian graphs, and assume that $V_1 = \{x_1, x_2, \ldots, x_n\}$ and $V_2 = \{y_1, y_2, \ldots, y_m\}$. Let P_1 and P_2 be Hamiltonian paths in H_1 and H_2, respectively. Permute labels, if needed, so that P_1 has initial point x_1 and terminal point x_n, and P_2 has initial point y_1 and terminal point y_m. Then it is easy to see that $P_1, x_n y_1, P_2, y_m x_1$ is a Hamiltonian cycle in $G = H_1 \square H_2$. □

Since Q_2 is Hamiltonian, it follows from the preceding result, and from the fact that $Q_k = Q_{k-1} \square Q_{k-1}$, that Q_k is Hamiltonian for all $k \geq 2$.